Construction of optimal codes in deletion and insertion metric

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Abstract

We improve Levenshtein's upper bound for the cardinality of a code of length four that is capable of correcting single deletions over an alphabet of even size. We also illustrate that the new upper bound is sharp. Furthermore we construct an optimal perfect code that is capable of correcting single deletions for the same parameters.

1 Introduction

Let $B_q = \{0, 1, \ldots, q-1\}$ be a set with q elements. B_q is referred to as an alphabet and its elements are referred to as letters. A sequence $x = (x_1, \ldots, x_n)$ of n letters of B_q is called a word, and the number n is called its length. Together with writing $x = (x_1, \ldots, x_n)$, we will also use the notation $x = x_1 \cdots x_n$. Let B_q^n be the set of words over B_q of length n, and define

$$B_q^* = \bigcup_{n=0}^{\infty} B_q^n.$$

The deletion and insertion distance $\rho(x, y)$, which was first introduced by Levenshtein [5], between two words x and y in B_q^* is defined by the minimum number of deletions and insertions of letters required to transform x into y. For example, let $x = 12243$ and $y = 14223$ be two words in B_5^5. Deleting the fourth letter of x and the second of y yields the identical words $x' = 1223$ and $y' = 1223$, respectively. Hence $\rho(x, y) = 2$. For a code $C \subseteq B_q^n$ with $|C| \geq 2$, we define

$$\rho(C) = \min \{ \rho(x, y) \mid x, y \in C, x \neq y \}.$$

Let $N(n, q, d) = \max \left\{ |C| \mid C \subseteq B_q^n, \rho(C) > 2d \right\}$. A code C in B_q^n is called an optimal code if $|C| = N(n, q, d)$. Levenshtein [6] estimated an upper bound of $N(n, q, 1)$ for any $n \geq 2$ and $q \geq 2$, namely,

$$N(n, q, 1) \leq \left\lfloor \frac{q^{n-1} + (n-2)q^{n-2} + q}{n} \right\rfloor. \tag{1}$$

The upper bound in (1) is sharp when $n = 3$. Indeed, a code that is capable of correcting single deletions whose cardinality meets the upper bound in (1) was constructed in [6] when $n = 3$ and $q \geq 2$ is an arbitrary integer. However this bound is not sharp when $n \geq 4$. In this paper, we improve Levenshtein's upper bound for the case in which $n = 4$ and q is even, and prove that the new upper bound is sharp.

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In Section 2, we will derive a new upper bound for the cardinality of a code that is capable of correcting single deletions when $n = 4$ and q is even by analyzing the deletion map.

In Section 3, we will construct a code in B_q^4 for $q \equiv 2$ or $4 \pmod 6$ that is capable of correcting single deletions whose cardinality meets the new upper bound derived in Section 2. We first introduce the concept of the step property for a Steiner quadruple system, and prove that there is a code in B_q^4 that is capable of correcting single deletions whose cardinality meets the upper bound derived in Section 2, under the assumption that there is a Steiner quadruple system on B_q that satisfies the step property. Then we show that there is a Steiner quadruple system on B_q that satisfies the step property for $q \equiv 2$ or $4 \pmod 6$.

In Section 4, we will construct a code in B_q^4 for $q \equiv 0 \pmod 6$ that is capable of correcting single deletions whose cardinality meets the upper bound derived in Section 2. Since there does not exist a Steiner quadruple system on an alphabet of this size, we will use a group divisible system. We divide our construction into two steps. In the first step, we prove that an optimal code exists for an alphabet of size $q = 6m$ where m is odd. In the next step, we prove that an optimal code exists for an alphabet of size $2q$ under the assumption that an optimal code exists for an alphabet of size q.

In Section 5, we modify our construction of optimal codes slightly, and construct an optimal perfect code in B_q^4 when q is even.

2 New upper bound of $N(4, q, 1)$ for q even

In this section, we improve Levenshtein's upper bound for the case in which $n = 4$ and q is even. Our result follows from an analysis of the deletion map as follows.

We begin with a simple observation. For any word x over B_q and any positive integer s, denote by $\lfloor x \rfloor_s$ the set of words obtained from x by deleting s of its letters. The multi-map $\lfloor\ \rfloor_s : B_q^n \to B_q^{n-s}$ which sends x to $\lfloor x \rfloor_s$ will be called an s-deletion map or simply a deletion map. For a subset $C \subseteq B_q^n$, denote by $\lfloor C \rfloor_s$ the set $\bigcup_{x \in C} \lfloor x \rfloor_s$. A set $C \subseteq B_q^n$ is called a code that is capable of correcting s deletions if all sets $\lfloor x \rfloor_s$ $(x \in C)$ do not pairwise intersect. It follows from the definition that a code C in B_q^n satisfies $\rho(C) > 2s$ if and only if C is capable of correcting s deletions.

The following lemma is an immediate consequence of this observation.

Lemma 2.1 (Substitution lemma)**.** *Let C be a code in B_q^n such that $\rho(C) > 2s$. If $x \in C$, $y \in B_q^n$, and $\lfloor y \rfloor_s \subseteq \lfloor x \rfloor_s$, then $\rho((C \setminus \{x\}) \cup \{y\}) > 2s$. More generally, if $A \subseteq C$, $B \subseteq B_q^n$, $\lfloor B \rfloor_s \subseteq \lfloor A \rfloor_s$, and $\rho(B) > 2s$, then $\rho((C \setminus A) \cup B) > 2s$.*

From now on, we will use C to denote a code in B_q^4 with $\rho(C) > 2$ and we define

$$C_i = \{x \in C \mid |\lfloor x \rfloor_1| = i\} \ (1 \leq i \leq 4).$$

Let a, b, c, and d be distinct elements in B_q. By the substitution lemma, we

may assume without changing the cardinality of C that

$$\begin{cases} x \in C_1 \Rightarrow x \text{ is of the type } (a,a,a,a), \\ x \in C_2 \Rightarrow x \text{ is of the type } (a,a,b,b), \\ x \in C_3 \Rightarrow x \text{ is of the type } (a,b,b,a), (a,a,b,c), (b,c,a,a), \text{ or } (a,b,b,c), \\ x \in C_4 \Rightarrow x \text{ is of the type } (a,b,c,d), (a,b,a,c), (a,b,c,a), \text{ or } (b,a,c,a). \end{cases}$$

In the case of $3 \le i \le 4$, we analyze further and define

$$\begin{cases} C_{3,1} = \{x \in C_3 \mid x \text{ is of the type } (a,a,b,c), (b,c,a,a), \text{ or } (a,b,b,c)\}, \\ C_{3,2} = \{x \in C_3 \mid x \text{ is of the type } (a,b,b,a)\}, \\ C_{4,1} = \{x \in C_4 \mid x \text{ is of the type } (a,b,c,d)\}, \\ C_{4,2} = \{x \in C_4 \mid x \text{ is of the type } (a,b,c,a)\}, \\ C_{4,3} = \{x \in C_4 \mid x \text{ is of the type } (a,b,a,c) \text{ or } (b,a,c,a)\}. \end{cases}$$

Consider the following subsets of B_q^3:

$$\begin{cases} U = \{y \in B_q^3 \mid y \text{ is of the type } (a,a,a)\}, \\ V = \{y \in B_q^3 \mid y \text{ is of the type } (a,a,b) \text{ or } (a,b,b)\}, \\ W = \{y \in B_q^3 \mid y \text{ is of the type } (a,b,a)\}, \\ Z = \{y \in B_q^3 \mid y \text{ is of the type } (a,b,c)\}. \end{cases}$$

Note that

$$\begin{cases} |U| = q, \ |V| = 2q(q-1), \\ |W| = q(q-1), \ |Z| = q(q-1)(q-2). \end{cases} \tag{2}$$

Table 1 counts the contribution of each codeword in C_i (or $C_{i,j}$) to U, V, W, and Z under the deletion map $\lfloor \ \rfloor_1 : B_q^4 \to B_q^3$.

	C_1	C_2	$C_{3,1}$	$C_{3,2}$	$C_{4,1}$	$C_{4,2}$	$C_{4,3}$
U	1	0	0	0	0	0	0
V	0	2	2	2	0	0	1
W	0	0	0	1	0	2	1
Z	0	0	1	0	4	2	2

Table 1: The contribution of each codeword in C_i (or $C_{i,j}$) to U, V, W, and Z

Let $X \in \{U,V,W,Z\}$ and define $X_i = X \cap \lfloor C_i \rfloor_1$ and $X_{i,j} = X \cap \lfloor C_{i,j} \rfloor_1$ (for example $V_{3,2} = V \cap \lfloor C_{3,2} \rfloor_1$). The following relations can be easily obtained from (2) and Table 1:

$$\begin{cases} |U_1| \le |U| = q, \\ |V_2| + |V_{3,1}| + |V_{3,2}| + |V_{4,3}| \le |V| = 2q(q-1), \\ |W_{3,2}| + |W_{4,2}| + |W_{4,3}| \le |W| = q(q-1), \\ |Z_{3,1}| + |Z_{4,1}| + |Z_{4,2}| + |Z_{4,3}| \le |Z| = q(q-1)(q-2), \\ |C_1| = |U_1|, \ |C_2| = \frac{1}{2}|V_2|, \ |C_{3,1}| = \frac{1}{2}|V_{3,1}| = |Z_{3,1}|, \ |C_{3,2}| = \frac{1}{2}|V_{3,2}| = |W_{3,2}|, \\ |C_{4,1}| = \frac{1}{4}|Z_{4,1}|, \ |C_{4,2}| = \frac{1}{2}|W_{4,2}| = \frac{1}{2}|Z_{4,2}|, \ |C_{4,3}| = |V_{4,3}| = |W_{4,3}| = \frac{1}{2}|Z_{4,3}|. \end{cases}$$

This information allows derivation of the first main result.

Theorem 2.2. *Let C be a code in B_q^4 with an even q and $\rho(C) > 2$. Then*

$$|C| \le \frac{q^2(q+2)}{4}.$$

Proof. It follows from the previous calculation that

$$\begin{array}{rcl} |C| & = & |C_1|+|C_2|+|C_{3,1}|+|C_{3,2}|+|C_{4,1}|+|C_{4,2}|+|C_{4,3}| \\ & = & |U_1|+\frac{1}{2}|V_2|+\frac{1}{2}|V_{3,1}|+\frac{1}{2}|V_{3,2}|+\frac{1}{4}|Z_{4,1}|+\frac{1}{2}|Z_{4,2}|+\frac{1}{2}|Z_{4,3}| \\ & = & |U_1|+\frac{1}{2}(|V_2|+|V_{3,1}|+|V_{3,2}|+|V_{4,3}|)-\frac{1}{2}|V_{4,3}| \\ & & +\frac{1}{4}(|Z_{3,1}|+|Z_{4,1}|+|Z_{4,2}|+|Z_{4,3}|)-\frac{1}{4}|Z_{3,1}|+\frac{1}{4}|Z_{4,2}|+\frac{1}{4}|Z_{4,3}| \\ & = & |U_1|+\frac{1}{2}(|V_2|+|V_{3,1}|+|V_{3,2}|+|V_{4,3}|)+\frac{1}{4}(|Z_{3,1}|+|Z_{4,1}|+|Z_{4,2}|+|Z_{4,3}|) \\ & & +\frac{1}{2}|C_{4,2}|-\frac{1}{4}|C_{3,1}| \\ & \leq & |U|+\frac{1}{2}|V|+\frac{1}{4}|Z|+\frac{1}{2}|C_{4,2}|-\frac{1}{4}|C_{3,1}|. \end{array}$$

Suppose that $(a,b_1,c_1,a)\in C_{4,2}$, $(a,b_2,c_2,a)\in C_{4,2}$, and $(a,b_1,c_1,a)\neq(a,b_2,c_2,a)$. From the condition $\rho(C)>2$, it follows that $\{b_1,c_1\}\cap\{b_2,c_2\}=\emptyset$. Therefore

$$|C_{4,2}|\leq q\left\lfloor\frac{q-1}{2}\right\rfloor=\frac{q(q-2)}{2}.$$

Since $|C_{3,1}|\geq 0$, the result follows. □

Remark 1. Theorem 2.2 improves the bound in (1) when q is even and it gives the same bound when q is odd. But this bound is not sharp when q is odd. Obtaining a sharp bound when q is odd seems to be a very difficult problem.

Remark 2. The proof of Theorem 2.2 can be used in the construction of optimal codes in the following way. Let C be a code in B_q^4 with $|C|=\frac{q^2(q+2)}{4}$. Then we should have

$$\begin{array}{l} |U_1|=|U|=q,\ |V_2|+|V_{3,1}|+|V_{3,2}|+|V_{4,3}|=|V|=2q(q-1),\ |C_{3,1}|=0, \\ |Z_{3,1}|+|Z_{4,1}|+|Z_{4,2}|+|Z_{4,3}|=|Z|=q(q-1)(q-2),\ |C_{4,2}|=\frac{q(q-2)}{2}. \end{array}$$

Since $|C_{3,1}|=0$, we have $|V_{3,1}|=|Z_{3,1}|=0$. From the fact that $|C_{4,2}|=\frac{q(q-2)}{2}$, we may assume that $|C_{3,2}|=|C_{4,3}|=0$. Then $|V_{3,2}|=|V_{4,3}|=|Z_{4,3}|=0$. In this case $|V_2|=|V|$. Hence

$$\begin{array}{l} |C_1|=q,\ |C_2|=q(q-1),\ |C_3|=0, \\ |C_{4,1}|=\frac{q(q-1)(q-2)-q(q-2)}{4},\ |C_{4,2}|=\frac{q(q-2)}{2}. \end{array}$$

This means that we can obtain an optimal code if we include every word of the types (a,a,a,a) and (a,a,b,b), the maximal number of words of the type (a,b,c,a), and generate words of the type (a,b,c,d) using all remaining words of length 3 of the type (a,b,c). Since $|U_1|=|U|$ and $|V_2|=|V|$ are always possible, to construct an optimal code we only need to consider combinations of elements in $W\cup Z$ that satisfy

$$|C_4|=\frac{q(q-1)(q-2)-q(q-2)}{4}+\frac{q(q-2)}{2}.$$

From these considerations, we can construct optimal codes in B_4^4 (Table 2) and B_6^4 (Table 3) which coincide with the construction in [4].

In the next two sections, we will construct an optimal code in B_q^4 when q is even. Let x be a word of length 4 which consists of pairwise distinct letters. The basic ingredients in our construction are the codes $\langle x\rangle_{A_q^4}$ generated by x in A_q^4 and $\langle x\rangle_{B_q^4}$ generated by x in B_q^4. These codes will be defined below. The code $\langle x\rangle_{A_q^4}$ was already used in [6]. The essence of our construction is to

0000	1111	2222	3333
0011	0022	0033	1100
1122	1133	2200	2211
2233	3300	3311	3322
0230	1231	2012	3013
0321	2103	1302	3120

Table 2: An optimal code in B_4^4

0230	1231	2012	3013	4014	5015
0450	1451	2452	3453	4234	5235
0251	1304	2053	3105	4035	5102
0342	1325	2140	3124	4120	5143
0431	1503	2413	3520	4215	5321
0524	1542	2504	3541	4302	5340

Table 3: The C_4 of an optimal code in B_6^4

replace $\langle x\rangle_{A_q^4}$ by $\langle x\rangle_{B_q^4}$ for some words such that our construction satisfies the conditions of Remark 2.

Let A_q^n be the set of all words in B_q^n that have pairwise distinct letters. For a word $x=(a_1,a_2,a_3,a_4)$ in A_q^4, the codes $\langle x\rangle_{A_q^4}$ and $\langle x\rangle_{B_q^4}$ are defined as follows:

$$\begin{aligned}
\langle x\rangle_{A_q^4} &= \{(a_1,a_2,a_3,a_4),(a_1,a_4,a_3,a_2),(a_2,a_4,a_1,a_3),\\
&\quad (a_3,a_4,a_1,a_2),(a_3,a_2,a_1,a_4),(a_4,a_2,a_3,a_1)\},\\
\langle x\rangle_{B_q^4} &= \Big(\langle x\rangle_{A_q^4}\setminus\{(a_1,a_2,a_3,a_4),(a_3,a_4,a_1,a_2)\}\Big)\\
&\quad \cup\{(a_1,a_3,a_4,a_1),(a_2,a_3,a_4,a_2),(a_3,a_1,a_2,a_3),(a_4,a_1,a_2,a_4)\}.
\end{aligned}$$

The following lemma which describes the basic properties of these codes under the deletion map can be easily verified.

Lemma 2.3. *Let $x=(a_1,a_2,a_3,a_4)$ and $y=(b_1,b_2,b_3,b_4)$ be distinct words of A_q^4, and $L(x)$ be the set of letters in x. Then*

(i) $\langle x\rangle_{A_q^4}$ and $\langle x\rangle_{B_q^4}$ are codes that are capable of correcting single deletions,

(ii) if $|L(x)\cap L(y)|\le 2$, then $\langle x\rangle_{A_q^4}\cup\langle y\rangle_{B_q^4}$ is a code that is capable of correcting single deletions,

(iii) if $\{a_1,a_2\}\cap\{b_1,b_2\}=\emptyset$ or $\{a_3,a_4\}\cap\{b_3,b_4\}=\emptyset$, then $\langle x\rangle_{B_q^4}\cup\langle y\rangle_{B_q^4}$ is a code that is capable of correcting single deletions.

3 Construction of optimal codes in B_q^4 for $q\equiv 2 \text{ or } 4 \pmod 6$

In this section, we will construct a code in B_q^4 for $q\equiv 2$ or $4 \pmod 6$ that is capable of correcting single deletions whose cardinality meets the upper bound that was established in Theorem 2.2. Our construction consists of two steps.

In the first step, we introduce the concept of the step property for a Steiner quadruple system, and prove that there is a code in B_q^4 that is capable of correcting single deletions whose cardinality meets the upper bound in Theorem 2.2, under the assumption that there is a Steiner quadruple system on B_q that satisfies the step property. In the next step, we follow the construction of Hanani [2] to show that there is a Steiner quadruple system on B_q that satisfies the step property for $q \equiv 2$ or $4 \pmod 6$.

When $q = 2$, we construct a code C in B_2^4 with codewords

$$C = \{(0,0,0,0),\ (1,1,1,1),\ (0,0,1,1),\ (1,1,0,0)\},$$

by following the method in Remark 2. Since $|C| = 4$, the code C should be an optimal code.

Let $SQS(q)$ denote a Steiner quadruple system on B_q. Recall that an $SQS(q)$ is a set of 4-element subsets of B_q, called quadruples, with the property that every 3-element subset of B_q is a subset of exactly one quadruple in the set. It is well known [7] that there is a Steiner quadruple system on B_q if and only if $q \equiv 2$ or $4 \pmod 6$. From now on, we assume that $q \equiv 2$ or $4 \pmod 6$ and that $q \geq 4$.

To construct optimal codes, we define the step property.

Definition 3.1 (The step property)**.** *Let $SQS(q)$ be a Steiner quadruple system on B_q and $L_0 < L_1 < \cdots < L_{q-1}$ be a total order on $B_q = \{L_0, L_1, \ldots, L_{q-1}\}$. Let $\{L_{2t},\ _{2t+1},\ L_a,\ L_b\}$ be a quadruple in $SQS(q)$. We say that $\{L_{2t},\ L_{2t+1},\ L_a,\ L_b\}$ satisfies the step property with respect to the given order if either $a < 2t$, $b < 2t$ or $2t+1 < a$, $2t+1 < b$. We also say that $SQS(q)$ satisfies the step property if every quadruple of the form $\{L_{2t},\ L_{2t+1},\ L_a,\ L_b\}$ satisfies the step property with respect to the given total order.*

If there is a Steiner quadruple system that has the step property, optimal codes can be constructed using the following theorem.

Theorem 3.2. *Suppose that there is an $SQS(q)$ that satisfies the step property. Then*

$$N(4, q, 1) = \frac{q^2(q+2)}{4}.$$

Proof. Suppose that there is an $SQS(q)$ that satisfies the step property. Without loss of generality, we may assume that $0 < 1 < \cdots < q-1$ is the total order which admits the step property for $SQS(q)$. Let $\phi : SQS(q) \to B_q^4$ be the map defined by $\phi(\{x, y, z, w\}) = (x, y, z, w)$ where $x < y < z < w$, and

$$S(q) = \{\{2t, 2t+1, a, b\} \in SQS(q) \mid 2t+1 < a,\ 2t+1 < b\}.$$

It follows from the definition of a Steiner quadruple system and Lemma 2.3 that the code

$$M = \left(\bigcup_{x \in \phi(SQS(q) \setminus S(q))} \langle x \rangle_{A_q^4} \right) \bigcup \left(\bigcup_{x \in \phi(S(q))} \langle x \rangle_{B_q^4} \right)$$

is capable of correcting single deletions. $|S(q)| = \frac{q(q-2)}{8}$, so

$$|M| = 6 \cdot \left\{ \frac{q(q-1)(q-2)}{24} - \frac{q(q-2)}{8} \right\} + 8 \cdot \frac{q(q-2)}{8} = \frac{q^2(q-2)}{4}.$$

We obtain a code C from M by adding all words of the types (a,a,a,a), and (a,a,b,b) $(a \neq b)$. It is easy to see that C is capable of correcting single deletions. Finally we have

$$|C| = q + q(q-1) + \frac{q^2(q-2)}{4} = \frac{q^2(q+2)}{4}.$$

□

Hanani [2] inductively constructed an $SQS(q)$ for $q \equiv 2$ or $4 \pmod 6$. Using Hanani's contruction, we will prove that there exists an $SQS(q)$ with the step property for all $q \equiv 2$ or $4 \pmod 6$.

In subsequence sections, the following definitions and notations will be used. For a natural number n, B_n (resp. $\bar{B}_n$) denotes the set $\{0,1,\dots,n-1\}$ (resp. $\{1,2,\dots,n\}$). We introduce two partitions of unordered pairs of the set B_{2m} [2].

We decompose the $m(2m-1)$ pairs $\{r,s\}$ from the set B_{2m} into $2m-1$ systems $P_\alpha(m)$ $(\alpha \in B_{2m-1})$, each containing m mutually disjoint pairs.

For $m \equiv 0 \pmod 2$ we form the systems $P_\alpha(m)$ as follows:

$$\begin{aligned}
P_{2\beta}(m) &= \{\{2a,\, 2a+2\beta+1\} \mid a \in B_m\} \ \left(\beta \in B_{\frac{m}{2}}\right), \\
P_{2\beta+1}(m) &= \{\{2a,\, 2a-2\beta-1\} \mid a \in B_m\} \ \left(\beta \in B_{\frac{m}{2}}\right), \\
P_{m+\gamma}(m) &= \left\{ \begin{array}{l} \{b,\, 2\gamma-b\} \mid b \in B_\gamma \\ \{c,\, 2m+2\gamma-c-2\} \mid 2\gamma+1 \le c \le m+\gamma-2 \\ \{2m-\frac{3}{2}-(-1)^\gamma\frac{1}{2}, \gamma\} \\ \{2m-\frac{3}{2}+(-1)^\gamma\frac{1}{2}m+\gamma-1\} \end{array} \right\} \ (\gamma \in B_{m-1}).
\end{aligned}$$

For $m \equiv 1 \pmod 2$ we let:

$$\begin{aligned}
P_{2\beta}(m) &= \{\{2a,\, 2a+2\beta+1\} \mid a \in B_m\} \ \left(\beta \in B_{\frac{m-1}{2}}\right), \\
P_{2\beta+1} &= \{\{2a,\, 2a-2\beta-1\} \mid a \in B_m\} \ \left(\beta \in B_{\frac{m-1}{2}}\right), \\
P_{m-1+\gamma}(m) &= \left\{ \begin{array}{l} \{b,\, 2\gamma-b\} \mid b \in B_\gamma \\ \{c, 2m+2\gamma-c\} \mid 2\gamma+1 \le c \le m+\gamma-1 \\ \{\gamma, m+\gamma\} \end{array} \right\} \ (\gamma \in B_m).
\end{aligned}$$

It can be easily verified that the pairs in every system are mutually disjoint and no pair appears twice. Because the number of pairs in the systems is $m(2m-1)$, it follows that every pair appears in some system.

We shall also need another decomposition of the $m(2m-1)$ pairs from B_{2m} into $2m$ systems $\bar{P}_\xi(m)$ $(\xi \in B_{2m})$ such that each of the m systems $\bar{P}_\eta(m)$ $(\eta \in B_m)$ should contain $m-1$ mutually disjoint pairs not containing the elements 2η and $2\eta+1$, and each of the other m systems should contain m mutually disjoint pairs. We shall form the system $\bar{P}_\xi(m)$ using the systems $P_\alpha(m)$.

If $m \equiv 0 \pmod 2$, it can easily be seen that

$$\begin{cases}
\{2\mu,\, 4\mu+1\} \in P_{2\mu}(m) \ \left(\mu \in B_{\frac{m}{2}}\right), \\
\{2m-2-2\mu,\, 2m-1-4\mu\} \in P_{2\mu-1}(m) \ \left(\mu \in \bar{B}_{\frac{m-2}{2}}\right), \\
\{2m-2,\, 0\} \in P_m(m), \\
\{2m-1,\, 1\} \in P_{m+1}(m).
\end{cases}$$

Clearly, these pairs are mutually disjoint. We remove them from their respective systems and use them to form a new system.

Performing the following permutation of the elements

$$\begin{pmatrix} 2\mu & 4\mu+1 & 2m-2-2\mu & 2m-1-4\mu & 2m-2 & 0 & 2m-1 & 1 \\ 4\mu & 4\mu+1 & 4\mu-2 & 4\mu-1 & 1 & 0 & 2m-1 & 2m-2 \end{pmatrix} \left(\mu \in \bar{B}_{\frac{m-2}{2}}\right),$$

we obtain the new systems $\bar{P}_\xi(m)$ by a suitable reordering of the systems.

If $m \equiv 1 \pmod 2$,

$$\begin{cases} \{2\mu,\, 4\mu+1\} \in P_{2\mu}(m) \ \left(\mu \in B_{\frac{m-1}{2}}\right), \\ \{2m-2-2\mu,\, 2m-3-4\mu\} \in P_{2\mu+1}(m) \ \left(\mu \in B_{\frac{m-1}{2}}\right), \\ \{m-1,\, 2m-1\} \in P_{2m-2}(m). \end{cases}$$

These pairs are again mutually disjoint.

By the permutation

$$\begin{pmatrix} 2\mu & 4\mu+1 & 2m-2-2\mu & 2m-3-4\mu & m-1 & 2m-1 \\ 2\mu & 4\mu+1 & 4\mu+2 & 4\mu+3 & 2m-2 & 2m-1 \end{pmatrix} \left(\mu \in B_{\frac{m-1}{2}}\right)$$

of the elements and using the same procedure as in the case $m \equiv 0$ (mod2) we obtain the systems $\bar{P}_\xi(m)$.

If a system $SQS(f)$ exists, we say that a quadruple system can be formed from B_f and write $f \in \mathcal{S}$. Similarly, if a system $SQS(f)$ with the step property exists, we denote the condition as $f \in \mathcal{SP}$.

If $f \in \mathcal{S}$, we shall use $\{x, y, z, t\} \in B_f$ to denote any quadruple in B_f, that is, an element of $SQS(f)$. If $f+1 \in \mathcal{S}$, then we assume that $SQS(f+1)$ is formed by the alphabet $B_f \cup \{A\}$ where A is an additional element. The quadruples that contain A will be denoted by $\{A, u, v, w\}$.

The following was inductively shown by Hanani [2].

Theorem 3.3. *If $q \equiv 2$ or 4 (mod 6), then $q \in \mathcal{S}$.*

Using a similar argument as in [2], we will prove the following.

Theorem 3.4. *Suppose that $q \geq 4$. If $q \equiv 2$ or 4 (mod 6), then $q \in \mathcal{SP}$.*

Proof. We will proceed by induction on q. Clearly, $SQS(4)$ has the step property, hence $4 \in \mathcal{SP}$. As an induction step, we will prove the following: Let $q \equiv 2$ or 4 (mod 6). If $f \in \mathcal{SP}$ for every $f < q$ satisfying $f \equiv 2$ or 4 (mod 6), then $q \in \mathcal{SP}$. The proof will be given separately for each of the following cases which evidently exhaust all the possibilities:

$$\begin{cases} q \equiv 4 \text{ or } 8 \pmod{12}, \\ q \equiv 4 \text{ or } 10 \pmod{18}, \\ q \equiv 34 \pmod{36}, \\ q \equiv 26 \pmod{36}, \\ q \equiv 2 \text{ or } 10 \pmod{24}\ (q > 2), \\ q \equiv 14 \text{ or } 38 \pmod{72}. \end{cases}$$

Case I : $q \equiv 4$ or 8 (mod 12).

Let $q = 2f$ where $f \equiv 2$ or 4 (mod 6). Since $f \in \mathcal{SP}$, there is an $SQS(f)$ with

the step property. Without loss of generality, we may assume that $0 < 1 < \cdots < f-1$ is the total order on B_f which admits the step property for $SQS(f)$. Let $N = \{(i,j) \mid i \in B_2,\, j \in B_f\}$ and $\{x,y,z,t\}$ be any element in $SQS(f)$. The following quadruples in N form an $SQS(q)$ [2] :

$$\begin{cases} Q_1 & : \quad \{(a_1,x),(a_2,y),(a_3,z),(a_4,t)\}\ (a_1+a_2+a_3+a_4 \equiv 0 \pmod 2), \\ Q_2 & : \quad \{(0,j),(0,j'),(1,j),(1,j')\}\ (j \neq j'). \end{cases}$$

We rename the letters of N as follows: for $j \in B_f$,

$$(0,j) \to j,\ (1,j) \to f+j. \tag{3}$$

Note that the solutions of $a_1+a_2+a_3+a_4 \equiv 0 \pmod 2$ are

$$(a_1,a_2,a_3,a_4) = \begin{cases} (0,0,0,0),\ (0,0,1,1),\ (0,1,0,1),\ (0,1,1,0), \\ (1,0,0,1),\ (1,0,1,0),\ (1,1,0,0),\ (1,1,1,1). \end{cases}$$

To investigate the step property on $SQS(q)$, we only need to consider the following quadruples in each type: for $t \in B_{\frac{t}{2}}$,

$$\begin{cases} Q_1 & : \quad \{(0,2t),\ (0,2t+1),\ (0,x),\ (0,y)\},\ \{(0,2t),\ (0,2t+1),\ (1,x),\ (1,y)\}, \\ & \quad\ \{(1,2t),\ (1,2t+1),\ (0,x),\ (0,y)\},\ \{(1,2t),\ (1,2t+1),\ (1,x),\ (1,y)\}, \\ Q_2 & : \quad \{(0,2t),\ (0,2t+1),\ (1,2t),\ (1,2t+1)\}. \end{cases}$$

Note that $\{2t,\ 2t+1,\ x,\ y\}$ is a quadruple in $SQS(f)$. Since $SQS(f)$ satisfies the step property with respect to the order $0 < 1 < \cdots < f-1$, it follows from (3) that $SQS(q)$ satisfies the step property. Therefore $q \in \mathcal{SP}$.

Case II : $q \equiv 4$ or $10 \pmod{18}$.

Let $q = 3f+1$ where $f \equiv 1$ or $3 \pmod 6$. Since $f+1 \equiv 2$ or $4 \pmod 6$, we obtain $f+1 \in \mathcal{SP}$ by induction. Hence we may assume that $SQS(f+1)$ satisfies the step property with respect to the order $0 < 1 < \cdots < f-1 < A$. In accordance with the notation in [2], we define $N = \{(i,j),\ A \mid i \in B_3,\ j \in B_f\}$. Also, let $\{x,y,z,t\}$ and $\{A,u,v,w\}$ be quadruples in B_f and $B_f \cup \{A\}$, respectively. Note that both of them are elements of $SQS(f+1)$. According to [2], the following quadruples in N form an $SQS(q)$:

$$\begin{cases} Q_1 & : \quad \{(a_1,x),(a_2,y),(a_3,z),(a_4,t)\}\ (a_1+a_2+a_3+a_4 \equiv 0 \pmod 3), \\ Q_2 & : \quad \{A,(b_1,u),(b_2,v),(b_3,w)\}\ (b_1+b_2+b_3 \equiv 0 \pmod 3), \\ Q_3 & : \quad \{(i,u),(i,v),(i+1,w),(i+2,w)\}, \\ Q_4 & : \quad \{(i,j),(i,j'),(i+1,j),(i+1,j')\}\ (j \neq j'), \\ Q_5 & : \quad \{A,(0,j),(1,j),(2,j)\}. \end{cases}$$

In cases Q_3 and Q_4, calculations are performed modulo 3.

We then rename letters in N as follows: for $t \in B_{\frac{f-1}{2}}$,

$$\begin{cases} (0,2t) \to 6t, & (0,2t+1) \to 6t+1, \\ (1,2t) \to 6t+2, & (1,2t+1) \to 6t+3, \\ (2,2t) \to 6t+4, & (2,2t+1) \to 6t+5 \\ (0,f-1) \to 3f-3, & (1,f-1) \to 3f-2, \\ (2,f-1) \to 3f-1, & A \to 3f. \end{cases} \tag{4}$$

To investigate the step property, we only need to consider the following quadruples of each type: for $t \in B_{\frac{f-1}{2}}$ and $i \in B_4$,

$$\begin{cases} Q_1 & : & \{(0,2t),(0,2t+1),(0,x),(0,y)\},\{(0,2t),(0,2t+1),(1,x),(2,y)\}, \\ & & \{(1,2t),(1,2t+1),(0,x),(1,y)\},\{(1,2t),(1,2t+1),(2,x),(2,y)\}, \\ & & \{(2,2t),(2,2t+1),(0,x),(2,y)\},\{(2,2t),(2,2t+1),(1,x),(1,y)\}, \\ Q_2 & : & \{(1,2t),(1,2t+1),(1,x),A\},\ \{(0,x),(1,y),(2,f-1),A\}, \\ Q_3 & : & \{(0,2t),(0,2t+1),(1,x),(2,x)\},\{(1,2t),(1,2t+1),(2,x),(0,x)\}, \\ & & \{(2,2t),(2,2t+1),(0,x),(1,x)\},\{(2,x),(2,y),(0,f-1),(1,f-1)\}, \\ Q_4 & : & \{(i,2t),(i,2t+1),(i+1,2t),(i+1,2t+1)\}, \\ Q_5 & : & \{A,(0,f-1),(1,f-1),(2,f-1)\}. \end{cases}$$

Note that $\{2t, 2t+1, a, b\}$, $\{2t, 2t+1, a, A\}$, and $\{a, b, f-1, A\}$ are quadruples in $SQS(f+1)$. Since $SQS(f+1)$ on $B_f \cup \{A\}$ satisfies the step property with respect to the order $0 < 1 < \cdots < f-1 < A$, it follows from (4) that $SQS(q)$ satisfies the step property. Therefore $q \in \mathcal{SP}$.

Case III : $q \equiv 34 \pmod{36}$.
Let $q = 3f + 4$ where $f \equiv 10 \pmod{12}$, and denote $f = 12k + 10$. By induction, we may assume that $f + 4 \in \mathcal{SP}$. In accordance with the notation in [2], we can assume that there is an $SQS(f + 4)$ on the alphabet $B_f \cup \{(A,0), (A,1), (A,2), (A,3)\}$. Defining $L_t = 2t$, $L_{\frac{f}{2}+t} = 2t+1$ $(t \in B_{\frac{f}{2}})$, we also assume that given $SQS(f + 4)$ satisfies the step property with respect to the order

$$L_0 < L_1 < \cdots < L_{f-1} < (A,0) < (A,1) < (A,2) < (A,3).$$

Let $N = \{(i,j),(A,h) \mid i \in B_3,\, j \in B_f,\, h \in B_4\}$ and $\{x, y, z, t\}$ be any quadruple in $SQS(f + 4)$. The following quadruples in N form an $SQS(q)$ [2]:

$$\begin{cases} Q_1 & : & \{(A,0),(A,1),(A,2),(A,3)\}, \\ Q_2 & : & \{(i,x),(i,y),(i,z),(i,t)\} \text{ (the quadruple in } Q_1 \text{ excluded)}, \\ Q_3 & : & \{(A,a_1),(0,a_2),(1,a_3),(2,a_4)\}(a_1+a_2+a_3+a_4 \equiv 0 \pmod{f}), \\ Q_4 & : & \{(i+2,b_3),(i,b_1+2k+1+i(4k+2)-d), \\ & & (i,b_1+2k+2+i(4k+2)+d),(i+1,b_2)\} \\ & & \text{where } b_1+b_2+b_3 \equiv 0 \pmod{f} \text{ and } d \in B_{2k+1}, \\ Q_5 & : & \{(i,r_\alpha),(i,s_\alpha),(i+1,r'_\alpha),(i+1,s'_\alpha)\} \text{ where } \{r_\alpha,s_\alpha\} \text{ and } \{r'_\alpha,s'_\alpha\} \\ & & \text{are (equal or different) pairs in } P_\alpha(6k+5)\ (4k+2 \le \alpha \le 12k+8). \end{cases}$$

In case Q_2, we define $(i,(A,h)) = (A,h)$ for all $i \in B_3$ and $h \in B_3$. For both cases Q_4 and Q_5, calculations are conducted modulo 3 and f for the first and second coordinates, respectively.

We rename letters in N as follows:

$$\begin{cases} (i, L_t) \to if + t \ (i \in B_3,\ t \in B_f), \\ (A,h) \to 3f + h \ (h \in B_4). \end{cases} \tag{5}$$

To investigate the step property, we first consider the following quadruples of each type except Q_5: for $t \in B_{\frac{f}{2}}$, $i \in B_3$, and $h \in B_2$,

$$\begin{cases} Q_1 & : & \{(A,0),(A,1),(A,2),(A,3)\}, \\ Q_2 & : & \{(i,L_{2t}),(i,L_{2t+1}),(i,x),(i,y)\},\{(i,x'),(i,y'),(A,2h),(A,2h+1)\}, \\ Q_3,\ Q_4 & : & \text{no quadruples to consider.} \end{cases}$$

Note that $\{(A,0),(A,1),(A,2),(A,3)\}$, $\{L_{2t},L_{2t+1},x,y\}$, and $\{x',y',(A,2h),(A,2h+1)\}$ are quadruples in $SQS(f+4)$. Since $SQS(f+4)$ satisfies the step property with respect to the given order, it can be easily checked by (5) that the quadruples above satisfy the step property.

In Q_5, the following quadruples should be considered: for $i \in B_3$,

$$\{(i,L_{2t}),(i,L_{2t+1}),(i+1,x),(i+1,y)\},\ \{(i,x'),(i,y'),(i+1,L_{2t}),(i+1,L_{2t+1})\},$$

where $\{2t,2t+1\}$ and $\{x,y\}$ are pairs in $P_\alpha(6k+5)$ $(4k+2 \le \alpha \le 12k+8)$. These quadruples also satisfy the step property. Therefore $q \in \mathcal{SP}$.

Case IV : $q \equiv 26 \pmod{36}$.
This case is similar to Case III. Let $q = 3f+2$ where $f \equiv 8 \pmod{12}$, and denote $f = 12k+8$. By induction, we have $f+2 \in \mathcal{SP}$. We may assume that there is an $SQS(f+2)$ on the alphabet $B_f \cup \{(A,0),(A,1)\}$. Defining $L_t = 2t$, $L_{\frac{f}{2}+t} = 2t+1$ $\left(t \in B_{\frac{f}{2}}\right)$, we also assume that the given $SQS(f+2)$ satisfies the step property with respect to the order

$$L_0 < L_1 < \cdots < L_{f-1} < (A,0) < (A,1).$$

Let $N = \{(i,j),\ (A,h) \mid i \in B_3,\ j \in B_f,\ h \in B_2\}$ and $\{x,y,z,t\}$ be any quadruple in $SQS(f+2)$. The following quadruples in N form an $SQS(q)$ [2]:

$$\begin{cases} Q_1 & : & \{(i,x),(i,y),(i,z),(i,t)\}, \\ Q_2 & : & \{(A,a_1),(0,a_2),(1,a_3),(2,a_4)\}\ (a_1+a_2+a_3+a_4 \equiv 0 \pmod{f}), \\ Q_3 & : & \{(i+2,b_3),(i,b_1+2k+1+i(4k+2)-d), \\ & & (i,b_1+2k+2+i(4k+2)+d),(i+1,b_2)\} \\ & & \text{where } b_1+b_2+b_3 \equiv 0 \pmod{f} \text{ and } d \in B_{2k+1}, \\ Q_4 & : & \{(i,r_\alpha),(i,s_\alpha),(i+1,r'_\alpha),(i+1,s'_\alpha)\} \text{ where } \{r_\alpha,s_\alpha\} \text{ and } \{r'_\alpha,s'_\alpha\} \\ & & \text{are (equal or different) pairs in } P_\alpha(6k+4)\ (4k+2 \le \alpha \le 12k+6). \end{cases}$$

In case Q_1, we define $(i,(A,h)) = (A,h)$ for all $i \in B_3$ and $h \in B_4$. For both cases Q_3 and Q_4, calculations are performed modulo 3 and f for the first and second coordinates, respectively.

We again rename letters in N as follows:

$$\begin{cases} (i,L_t) \to if+t\ (i \in B_3,\ t \in B_f), \\ (A,h) \to 3f+h\ (h \in B_2). \end{cases} \tag{6}$$

To investigate the step property, we will first consider the following quadruples of each type except Q_4 : for $t \in B_{\frac{f}{2}}$ and $i \in B_3$,

$$\begin{cases} Q_1 & : & \{(i,L_{2t}),(i,L_{2t+1}),(i,a),(i,b)\},\{(i,a'),(i,b'),(A,0),(A,1)\}, \\ Q_2,\ Q_3 & : & \text{no quadruples to consider.} \end{cases}$$

Note that $\{L_{2t},L_{2t+1},a,b\}$ and $\{a',b',(A,0),(A,1)\}$ are quadruples in $SQS(f+2)$. Since $SQS(f+2)$ satisfies the step property with respect to the order above, it can be easily checked by (6) that the quadruples above satisfy the step property.

In case Q_4, the following quadruples should be considered: for $i \in B_3$,

$$\{(i,L_{2t}),(i,L_{2t+1}),(i+1,a),(i+1,b)\},\ \{(i,a),(i,b),(i+1,L_{2t}),(i+1,L_{2t+1})\},$$

where $\{L_{2t}, L_{2t+1}\}$ and $\{a, b\}$ are pairs in $P_\alpha(6k+4)$ $(4k+2 \le \alpha \le 12k+6)$. These quadruples also satisfy the step property. Therefore $q \in \mathcal{SP}$.

Case V : $q \equiv 2$ or $10 \pmod{24}$ $(q > 2)$.

Let $q = 4f + 2$ where $f \equiv 0$ or $2 \pmod 6$ $(f > 0)$, and denote $f = 2k$. By induction, we have $f + 2 \in \mathcal{SP}$. Hence we may assume that $SQS(f+2)$ on $B_f \cup \{(A,0),(A,1)\}$ satisfies the step property with respect to the order

$$0 < 1 < \cdots < f - 1 < (A,0) < (A,1).$$

Let $N = \{(h,i,j),\ (A,l) \mid h \in B_2,\ i \in B_2,\ j \in B_f,\ l \in B_2\}$ and $\{x,y,z,t\}$ be any quadruple in $SQS(f+2)$. The following quadruples in N form an $SQS(q)$ [2]: assuming that $c_1 + c_2 + c_3 \equiv 0 \pmod k$ and $\epsilon \in B_2$,

$$\begin{cases} Q_1 & : & \{(h,i,x),(h,i,y),(h,i,z),(h,i,t)\}, \\ Q_2 & : & \{(A,l),(0,0,2c_1),(0,1,2c_2-\epsilon),(1,\epsilon,2c_3+l)\}, \\ Q_3 & : & \{(A,l),(0,0,2c_1+1),(0,1,2c_2-1-\epsilon),(1,\epsilon,2c_3+1-l)\}, \\ Q_4 & : & \{(A,l),(1,0,2c_1),(1,1,2c_2-\epsilon),(0,\epsilon,2c_3+1-l)\}, \\ Q_5 & : & \{(A,l),(1,0,2c_1+1),(1,1,2c_2-1-\epsilon),(0,\epsilon,2c_3+l)\}, \\ Q_6 & : & \{(h,0,2c_1+\epsilon),(h,1,2c_2-\epsilon),(h+1,0,\bar{r}_{c_3}),(h+1,0,\bar{s}_{c_3})\} \\ & & \text{where } \{\bar{r}_{c_3},\bar{s}_{c_3}\} \text{ are pairs in } \bar{P}_{c_3}(k), \\ Q_7 & : & \{(h,0,2c_1-1+\epsilon),(h,1,2c_2-\epsilon),(h+1,1,\bar{r}_{c_3}),(h+1,1,\bar{s}_{c_3})\}, \\ Q_8 & : & \{(h,0,2c_1+\epsilon),(h,1,2c_2-\epsilon),(h+1,1,\bar{r}_{k+c_3}),(h+1,1,\bar{s}_{k+c_3})\}, \\ Q_9 & : & \{(h,0,2c_1-1+\epsilon),(h,1,2c_2-\epsilon),(h+1,0,\bar{r}_{k+c_3}),(h+1,0,\bar{s}_{k+c_3})\}, \\ Q_{10} & : & \{(h,0,r_\alpha),(h,0,s_\alpha),(h,1,r'_{\alpha'}),(h,1,s'_\alpha)\} \text{ where } \{r_\alpha,s_\alpha\} \text{ and } \{r'_{\alpha'},s'_\alpha\} \\ & & \text{are (equal or different) pairs in } P_\alpha(k)\ (\alpha \in B_{f-1}). \end{cases}$$

Note that if a quadruple in Q_1 contains an element of the form $(h,i,(A,l))$, we simply denote it by (A,l). For each case, calculations are conducted modulo 2 and f for the first and third coordinates, respectively.

We rename letters in N as follows: for $j \in B_f$,

$$\begin{cases} (0,0,j) \to j, & (0,1,j) \to f+j, \\ (1,0,j) \to 2f+j, & (1,1,j) \to 3f+j, \\ (A,0) \to 4f, & (A,1) \to 4f+1. \end{cases} \tag{7}$$

To investigate the step property, the following quadruples of each type should be considered, except $Q_6 - Q_{10}$: for $t \in B_{\frac{f}{2}}$, $h \in B_2$, and $i \in B_2$,

$$\begin{cases} Q_1 & : & \{(h,i,2t),(h,i,2t+1),(h,i,a),(h,i,b)\},\ \{(A,0),(A,1),(h,i,a),(h,i,b)\}, \\ Q_2 - Q_5 & : & \text{no quadruples to consider.} \end{cases}$$

In case Q_1, the quadruples $\{2t, 2t+1, a, b\}$ and $\{(A,0),(A,1),a,b\}$ are in $SQS(f+2)$. Since $SQS(f+2)$ satisfies the step property with respect to the given order, it can be easily checked by (7) that the quadruples above satisfy the step property.

In Q_6, the following quadruples are to be considered: for $h \in B_2$ and $\epsilon \in B_2$,

$$\{(h,0,2c_1+\epsilon),(h,1,2c_2-\epsilon),(h+1,0,2t),(h+1,0,2t+1)\}.$$

These quadruples also satisfy the step property. Cases $Q_7 - Q_{10}$ are similar to Q_6. As a result, $q \in \mathcal{SP}$.

{0,1,2,4}	{13,9,12,11}	{12,1,4,3}	{12,8,5,1}	{5,6,10,11}	{3,5,13,8}	{4,5,7,8}
{7,10,13,12}	{0,8,9,11}	{13,2,5,4}	{13,9,6,2}	{6,0,11,12}	{4,6,7,9}	{9,12,4,0}
{1,2,3,5}	{7,3,6,5}	{3,4,10,11}	{7,10,0,3}	{0,1,12,13}	{5,0,8,10}	{10,13,5,1}
{2,3,4,6}	{1,9,10,12}	{9,12,2,5}	{8,11,1,4}	{1,2,13,7}	{6,1,9,11}	{11,7,6,2}
{3,4,5,0}	{2,10,11,13}	{6,1,13,8}	{0,2,7,9}	{2,3,7,8}	{0,2,10,12}	{12,8,0,3}
{4,5,6,1}	{3,11,12,7}	{4,5,11,12}	{1,3,8,10}	{3,4,8,9}	{5,6,8,9}	{13,9,1,4}
{5,6,0,2}	{4,12,13,8}	{5,6,12,13}	{2,4,9,11}	{13,9,0,3}	{8,11,3,6}	{7,10,2,5}
{6,0,1,3}	{5,13,7,9}	{6,0,13,7}	{3,5,10,12}	{7,10,1,4}	{3,5,9,11}	{4,6,10,12}
{8,11,7,13}	{6,7,8,10}	{0,1,7,8}	{4,6,11,13}	{8,11,2,5}	{6,0,9,10}	{5,0,11,13}
{9,12,8,7}	{8,4,0,6}	{1,2,8,9}	{5,0,12,7}	{9,12,3,6}	{0,1,10,11}	{6,1,12,7}
{10,13,9,8}	{9,5,1,0}	{2,3,9,10}	{4,5,9,10}	{10,13,4,0}	{1,2,11,12}	{0,2,13,8}
{11,7,10,9}	{10,6,2,1}	{10,13,3,6}	{12,8,6,2}	{11,7,5,1}	{2,3,12,13}	{1,3,7,9}
{12,8,11,10}	{11,0,3,2}	{11,7,4,0}	{1,3,11,13}	{2,4,12,7}	{3,4,13,7}	{2,4,8,10}

Table 4: An SQS(14)

Case VI : $q \equiv 14$ or $38 \pmod{72}$.
Let $q = 12f + 2$ where $f \equiv 1$ or $3 \pmod 6$. As the first step, we will prove that $14 \in \mathcal{S}$ and $38 \in \mathcal{S}$. According to [1], we can construct an $SQS(14)$ (Table 4).

Defining $N' = \{(i,j),\, (A,h) \mid i \in B_3,\, j \in B_{12},\, h \in B_2\}$ to be a set of 38 elements, we will show that $38 \in \mathcal{S}$. Let $\{x', y', z', t'\}$ be any quadruple in $SQS(14)$. The following quadruples in N' form an $SQS(38)$ [2]: assuming that $b_1 + b_2 + b_3 \equiv 0 \pmod{12}$, $\epsilon \in B_2$, $g \in B_6$, and $e \in B_4$,

$$\left\{\begin{array}{lcl}
Q_1 & : & \{(i,x'),(i,y'),(i,z'),(i,t')\},\\
Q_2 & : & \{(A,h),(0,b_1),(1,b_2),(2,b_3+3h)\},\\
Q_3 & : & \{(i,b_1+4+i),(i,b_1+7+i),(i+1,b_2),(i+2,b_3)\},\\
Q_4 & : & \{(i,j),(i+1,j+6\epsilon),(i+2,6\epsilon-2j+1),(i+2,6\epsilon-2j-1)\},\\
Q_5 & : & \{(i,j),(i+1,j+6\epsilon),(i+2,6\epsilon-2j+2),(i+2,6\epsilon-2j-2)\},\\
Q_6 & : & \{(i,j),(i+1,j+6\epsilon-3),(i+2,6\epsilon-2j+1),(i+2,6\epsilon-2j+2)\},\\
Q_7 & : & \{(i,j),(i+1,j+6\epsilon+3),(i+2,6\epsilon-2j-1),(i+2,6\epsilon-2j-2)\},\\
Q_8 & : & \{(i,j),(i,j+6),(i+1,j+3\epsilon),(i+1,j+6+3\epsilon)\},\\
Q_9 & : & \{(i,2g+3\epsilon),(i,2g+6+3\epsilon),(i',2g+1),(i',2g+5)\}\ (i' \neq i),\\
Q_{10} & : & \{(i,2g+3\epsilon),(i,2g+6+3\epsilon),(i',2g+2),(i',2g+4)\},\\
Q_{11} & : & \{(i,j),(i,j+1),(i+1,j+3e),(i+1,j+3e+1)\},\\
Q_{12} & : & \{(i,j),(i,j+2),(i+1,j+3e),(i+1,j+3e+2)\},\\
Q_{13} & : & \{(i,j),(i,j+4),(i+1,j+3e),(i+1,j+3e+4)\},\\
Q_{14} & : & \{(i,r_\alpha),(i,s_\alpha),(i',r'_\alpha),(i',s'_\alpha)\} \text{ where } \{r_\alpha,s_\alpha\} \text{ and } \{r'_\alpha,s'_\alpha\}\\
 & & \text{are (equal or different) pairs in } P_\alpha(6)\ (4 \le \alpha \le 5).
\end{array}\right.$$

In case Q_1, we define $(i,(A,h)) = (A,h)$ for all $i \in B_3$ and $h \in B_2$. For each case, calculations are conducted modulo 3 and 12 for the first and second coordinates, respectively.

Now we construct $SQS(q)$ for $q \equiv 14$ or $38 \pmod{72}$, that is, $f \equiv 1$ or $3 \pmod 6$. Since $f \equiv 1$ or $3 \pmod 6$, we can assume that $f + 1 \in \mathcal{S}$. In this case, the alphabet of $SQS(f+1)$ is $B_f \cup \{A\}$ as we assumed. Let $N = \{(i,j),\, (A,h) \mid i \in B_f,\, j \in B_{12},\, h \in B_2\}$ be a set of $12f + 2$ elements and $\{A, u, v, w\}$ be any quadruple in $SQS(f+1)$ that contains A. The following quadruples from N

form an $SQS(q)$ [2]:

$$\begin{cases} R_1 \quad : \quad \{(i,x'),(i,y'),(i,z'),(i,t')\}, \\ R_2 \quad : \quad \begin{cases} \{(A,h),(u,b_1),(v,b_2),(w,b_3+3h)\} \ (b_1+b_2+b_3 \equiv 0 \pmod{12}), \\ \{(u,\alpha_1),(v,\alpha_2),(w,\alpha_3),(w,\alpha_4)\}, \\ \{(i,\beta_1),(i,\beta_2),(i',\beta_3),(i',\beta_4)\} \ \ (\{i',\, i\} \subset \{u,\, v,\, w\} \text{ and } i' \neq i)\,. \end{cases} \end{cases}$$

In case R_2, α_ν and β_ν ($\nu \in \bar{B}_4$) are to be replaced by the second indices of $Q_3 - Q_{14}$, corresponding to the first indices 0, 1, 2 for u, v, w, respectively. Note that i and i' define uniquely a $\{u, v, w\}$ in which they are contained. Therefore they may be considered as two indices from $\{u, v, w\}$.

$$R_3 \quad : \quad \{(x,a_1),(y,a_2),(z,a_3),(t,a_4)\} \ (a_1+a_2+a_3+a_4 \equiv 0 \pmod{12}).$$

From now on, we will show that $q \in \mathcal{SP}$ for $q \equiv 14$ or $38 \pmod{72}$. In the case that $q = 14$, we rename the letters of B_{14} in Table 4 as follows:

$$\begin{cases} t \to L_{2t} \ (t \in B_6), \\ t \to L_{2t-13} \ (7 \le t \le 12), \\ 6+7i \to L_{i+12} = (A,i) \ (i \in B_2) \end{cases}$$

Assuming $L_0 < L_1 < \cdots < L_{13}$, the following quadruples satisfy the step property:

	{0,7,1,8},	{0,7,2,9},	{0,7,3,10},	{0,7,4,11},	{0,7,5,12},	{0,7,6,13},
{1,8,0,7},		{1,8,2,9},	{1,8,3,10},	{1,8,4,11},	{1,8,5,12},	{1,8,6,13},
{2,9,0,7},	{2,9,1,8},		{2,9,3,10},	{2,9,4,11},	{2,9,5,12},	{2,9,6,13},
{3,10,0,7},	{3,10,1,8},	{3,10,2,9},		{3,10,4,11},	{3,10,5,12},	{3,10,6,13},
{4,11,0,7},	{4,11,1,8},	{4,11,2,9},	{4,11,3,10},		{4,11,5,12},	{4,11,6,13},
{5,12,0,7},	{5,12,1,8},	{5,12,2,9},	{5,12,3,10},	{5,12,4,11},		{5,12,6,13},
{6,13,0,7},	{6,13,1,8},	{6,13,2,9},	{6,13,3,10},	{6,13,4,11},	{6,13,5,12}.	

This shows that $14 \in \mathcal{SP}$.

In the case that $q = 38$, we rename letters of N' as follows:

$$\begin{cases} (i,L_t) \to 12i+t \ (i \in B_3,\, t \in B_{12}), \\ (A,h) \to 36+h \ (h \in B_2). \end{cases} \tag{8}$$

To investigate the step property of $SQS(38)$, we will consider quadruples of each type.

In case Q_1, the following quadruples should be considered: for $i \in B_3$,

$$\{(i,L_{2t}),(i,L_{2t+1}),(i,a),(i,b)\},\ \{(i,a),(i,b),(A,0),(A,1)\}.$$

Since $\{L_{2t}, L_{2t+1}, a, b\}$ and $\{a, b, (A,0), (A,1)\}$ are in $SQS(14)$ and satisfy the step property, these quadruples also satisfy the step property by (8).

In cases $Q_2 - Q_{13}$, there are no quadruples to consider.

In case Q_{14}, the following quadruples should be considered:

$$\{(i,L_{2t}),(i,L_{2t+1}),(i',a),(i',b)\}.$$

By (8), these quadruples satisfy the step property. Therefore $38 \in \mathcal{SP}$.

Finally, consider general cases. First, rename letters in N as follows:

$$\begin{cases} (i,L_t) \to 12i+t \ (i \in B_f,\, t \in B_{12}), \\ (A,h) \to 12f+h \ (h \in B_2). \end{cases} \tag{9}$$

From now on, we will investigate the step property of quadruples of each type.

$R_1)$ $\{(i, L_{2t}), (i, L_{2t+1}), (i, a), (i, b)\}$, $\{(i, a), (i, b), (A, 0), (A, 1)\}$.
Since $\{L_{2t}, L_{2t+1}, a, b\}$ and $\{a, b, (A, 0), (A, 1)\}$ satisfy the step property, these quadruples also satisfy the step property by (9).

$R_2)$ $\{(i, L_{2t}), (i, L_{2t+1}), (i', a), (i', b)\}$.
By the step property of the constructed $SQS(38)$, these quadruples satisfy the step property by (9).

$R_3)$ In this case, there are no quadruples to consider.
Therefore $q \in \mathcal{SP}$ $\square$

We constructed an optimal code in B_2^4, so if we use Theorems 3.2 and 3.4, we can deduce the following theorem.

Theorem 3.5. *If $q \equiv 2$ or $4\,(mod\ 6)$, then $N(4, q, 1) = \frac{q^2(q+2)}{4}$.*

Remark 3. The authors of [10] informed us that every Steiner quadruple system in [10] satisfies the step property. It is not known whether every Steiner quadruple system satisfies the step property.

4 Construction of optimal codes in B_q^4 for $q \equiv 0\,(\mathbf{mod}\,6)$

In this section, we will construct a code in B_q^4 for $q \equiv 0\,(\mathrm{mod}\,6)$ that is capable of correcting single deletions whose cardinality meets the upper bound that was established in Theorem 2.2. Since there dose not exist a Steiner quadruple system for any alphabet of this size, we will use a group divisible system. We divide our construction into two steps. In the first step, we prove that an optimal code exists over an alphabet of size $q = 6m$, where m is odd. In the next step, we prove that an optimal code exists over an alphabet of size $2q$ under the assumption that an optimal code exists over an alphabet of size q.

We begin with the definition of a group divisible system. By an r-subset of a set X, we mean a subset of X with r elements.

Definition 4.1 ([8])**.** *Let m and r be positive integers. Let $\mathcal{T} = \{T_1, T_2, \ldots, T_m\}$ be a collection of disjoint r-sets whose union is T. An (m, r, k, b) group divisible system or a $G(m, r, k, b)$ system on $\mathcal{T}$ is a collection $\mathcal{B} = \{K_1, K_2, \ldots, K_u\}$ of k-subsets of T such that every b-subset in T is either contained in T_i for some $i \in \bar{B}_m$ or it is contained in a unique k-subset in $\mathcal{B}$ but not both.*

By a simple counting argument,

$$|G(m, r, k, b)| = \frac{\binom{mr}{b} - m\binom{r}{b}}{\binom{k}{b}}.$$

The following result was proved by Mills [8].

Theorem 4.2. *A $G(m, 6, 4, 3)$ system exists for every positive integer m.*

Because we will use a subfamily $\mathcal{A}_1$ of a $G(m,6,4,3)$ system in our construction of an optimal code, we briefly review the proof of Theorem 4.2 in [8], which introduces $\mathcal{A}_1$ when m is odd. Let m be odd. For $i \in \bar{B}_m$ let T_i be the set of ordered pairs (i,α) with $\alpha \in B_6$ and T be the union of these T_i. We partition the elements of B_6 into pairs in three ways:

$$\begin{cases} P_1 = \{(0,3),(1,5),(2,4)\}, \\ P_2 = \{(1,4),(2,0),(3,5)\}, \\ P_3 = \{(2,5),(3,1),(4,0)\}. \end{cases}$$

For any ordered pair (w,x) $(w<x)$ in $\bar{B}_m$ and any $\lambda \in \bar{B}_3$, we form the nine quadruples

$$\{(w,\alpha),(w,\beta),(x,\gamma),(x,\delta)\},$$

where $(\alpha,\beta) \in P_\lambda$ and $(\gamma,\delta) \in P_\lambda$. This gives a collection of $\frac{27m(m-1)}{2}$ quadruples, denoted by $\mathcal{A}$, which is contained in $G(m,6,4,3)$. Among the quadruples in $\mathcal{A}$, choose those with $\lambda = 1$ and denote them by $\mathcal{A}_1$. Note that $|\mathcal{A}_1| = 9\binom{m}{2}$.

Theorem 4.3. *$N(4,q,1) = \frac{q^2(q+2)}{4}$ for $q = 6m$ where m is odd.*

Proof. We retain the notation used in the preceding discussion throughout this proof. Let m be odd and consider $G(m,6,4,3)$ on $\mathcal{T} = \{T_1,T_2,\ldots,T_m\}$. Recall that our goal is to construct an optimal code in B_q^4 where $q = |T| = 6m$. Define an order on T as follows:

$$\begin{cases} (i,\alpha) < (j,\alpha) \ (i<j), \\ (i,\alpha) < (j,\beta) \ (\alpha<\beta). \end{cases}$$

To each quadruple $\{x,y,z,w\}$ in $G(m,6,4,3)$ with $x<y<z<w$, we associate the word (x,y,z,w). From now on, we consider the quadruples in $G(m,6,4,3)$ as words of length 4 defined as above.

Let

$$M = \left(\bigcup_{x \in \mathcal{A}_1} \langle x \rangle_{B_q^4} \right) \bigcup \left(\bigcup_{x \in G(m,6,4,3) \setminus \mathcal{A}_1} \langle x \rangle_{A_q^4} \right).$$

Note that for any two distinct elements $a = (a_1,a_2,a_3,a_4)$ and $b = (b_1,b_2,b_3,b_4)$ of $\mathcal{A}_1$, if $|L(a) \cap L(b)| = 2$, then either $\{a_1,a_2\} = \{b_1,b_2\}$ or $\{a_3,a_4\} = \{b_3,b_4\}$. From the structure of $\mathcal{A}_1$, the definition of $G(m,6,4,3)$, and Lemma 2.3, we deduce that M is a code in B_q^4 that is capable of correcting single deletions. Since $|\mathcal{A}_1| = 9\binom{m}{2}$ and $|G(m,6,4,3)| = \frac{6m(6m-1)(6m-2)}{24} - 5m$, it follows that

$$|M| = 8|\mathcal{A}_1| + 6|G(m,6,4,3) \setminus \mathcal{A}_1| = 54m^3 - 18m^2 - 36m.$$

Since $N(4,6,1) = 72$ (cf. Table 3) and $|T_i| = 6$ for each $i \in \bar{B}_m$, we can choose an optimal code C_i over T_i of cardinality 72 that is capable of correcting single deletions. Let

$$C = \left(\bigcup_{i=1}^{m} C_i \right) \bigcup M \bigcup \{(a,a,b,b) \mid a \in T_i, b \in T_j, i \neq j\}.$$

It is easy to check that C is a code in B_q^4 that is capable of correcting single deletions. From a simple calculation

$$\begin{aligned}|C| &= m|C_1| + |M| + 2 \times 36\binom{m}{2} \\ &= 72m + 54m^3 - 18m^2 - 36m + 36m(m-1) \\ &= 54m^3 + 18m^2 = \frac{q^2(q+2)}{4}.\end{aligned}$$

This proves the theorem. □

The following lemma, originally due to Reiss [9], constructs the systems that are equivalent to the systems $P_\alpha(m)$. Because we need some terminology used in [3] to construct optimal codes, we borrow a sketch of the proof from [3].

Lemma 4.4 (Reiss)**.** *The $n(2n-1)$ pairs of $2n$ elements can be partitioned into $2n-1$ sets S_1, S_2, ..., S_{2n-1} such that each set contains n disjoint pairs.*

Proof. Let $l_{ij} = i + j - 1 \pmod{2n-1}$ where $i \in \bar{B}_{2n-1}$, $j \in \bar{B}_{2n-1}$, and $l_{ij} \in \bar{B}_{2n-1}$, and define $l_{i,2n} = l_{ii}$ $(i \in \bar{B}_{2n-1})$. Take

$$S_q = \{(i,j) \mid i < j \text{ and } l_{ij} = q\} \ (q \in \bar{B}_{2n-1}).$$

Then these sets S_q $(q \in \bar{B}_{2n-1})$ have the desired property. □

Theorem 4.5. *If $N(4,q,1) = \frac{q^2(q+2)}{4}$ for $q = 6m$, then $N(4,2q,1) = \frac{(2q)^2(2q+2)}{4}$.*

Proof. Suppose that $N(4,q,1) = \frac{q^2(q+2)}{4}$ for $q = 6m$. Let $\bar{B}_q(\alpha) = \{(a,\alpha) \mid a \in \bar{B}_q\}$ $(\alpha \in \bar{B}_2)$. We will construct a code in $\bar{B}_q(1) \cup \bar{B}_q(2)$ that is capable of correcting single deletions with cardinality $\frac{(2q)^2(2q+2)}{4}$. By applying Lemma 4.4 with $n = 3m$, the pairs of elements of $\bar{B}_q(\alpha)$ can be partitioned into the sets S_1^α, S_2^α, ..., S_{6m-1}^α, namely

$$\text{the set of pairs of } \bar{B}_q(\alpha) = S_1^\alpha \cup S_2^\alpha \cup \cdots \cup S_{6m-1}^\alpha \ (\alpha \in \bar{B}_2),$$

where $S_l^\alpha = \{((a,\alpha),(b,\alpha)) \mid a < b, a + b - 1 = l \pmod{6m-1}\}$ for $\alpha \in \bar{B}_2$ and $l \in \bar{B}_{6m-1}$. Note that $|S_l^\alpha| = 3m = \frac{q}{2}$ for all α and l. Consider the set $S^{1,2}$ of ordered pairs defined as follows:

$$S^{1,2} = \left\{(x,y) \mid x \in S_i^1, y \in S_i^2, i \in \bar{B}_{6m-1}\right\} = \bigcup_{i=1}^{6m-1} \left(S_i^1 \times S_i^2\right).$$

Since x and y are pairs of elements in $\bar{B}_q(\alpha)$ $\left(\alpha \in \bar{B}_2\right)$, $S^{1,2}$ is a set of quadruples of elements in $\bar{B}_q(1) \cup \bar{B}_q(2)$ whose cardinality is $(6m-1)|S_1^1||S_1^2| = 9m^2(6m-1) = \frac{q^2(q-1)}{2}$. Furthermore, $S^{1,2}$ has the following property: every triple $\{(c_1,i),(c_2,j),(c_3,k)\}$ with elements from $\bar{B}_q(1) \cup \bar{B}_q(2)$, except triples with element from only one of $\bar{B}_q(1)$ or $\bar{B}_q(2)$, belongs to a unique quadruple in $S^{1,2}$.

We define an order on $\bar{B}_q(1) \cup \bar{B}_q(2)$ as follows:

$$\begin{cases}(a,\alpha) < (b,\alpha) \text{ if and only if } a < b, \\ (a,1) < (b,2) \text{ for all } a, b.\end{cases}$$

To each quadruple $x = \{(a,1),(b,1),(c,2),(d,2)\}$ in $S^{1,2}$ where $((a,1),(b,1)) \in S_i^1$ and $((c,2),(d,2)) \in S_i^2$, we associate a word (a_1,a_2,a_3,a_4) of length 4 such that $a_1 < a_2 < a_3 < a_4$ and $a_i \in \{(a,1),(b,1),(c,2),(d,2)\}$. From now on, we consider the quadruples in $S^{1,2}$ as words of length 4 over $\bar{B}_q(1) \cup \bar{B}_q(2)$ defined as above. Let

$$M = \left(\bigcup_{x \in S_1^1 \times S_1^2} \langle x \rangle_{B_{2q}^4} \right) \bigcup \left(\bigcup_{x \in S^{1,2} \setminus (S_1^1 \times S_1^2)} \langle x \rangle_{A_{2q}^4} \right).$$

From the construction of $S^{1,2}$ and Lemma 2.3, M is a code in $\bar{B}_q(1) \cup \bar{B}_q(2)$ that is capable of correcting single deletions. Note that

$$|M| = 8 \left|S_1^1\right| \left|S_1^2\right| + 6 \left(\left|S^{1,2}\right| - \left|S_1^1\right| \left|S_1^2\right| \right) = \frac{2q^2(3q-2)}{4}.$$

Since $N(4,q,1) = \frac{q^2(q+2)}{4}$ for $q = 6m$, there exists a code C^α in $\bar{B}_q(\alpha)$ $(\alpha \in \bar{B}_2)$ that is capable of correcting single deletions with cardinality $|C^\alpha| = N(4,q,1) = \frac{q^2(q+2)}{4}$ $(\alpha \in B_2)$.

Finally, let

$$C = C^1 \cup C^2 \cup M \cup \{(a,a,b,b),\ (b,b,a,a) \mid a \in \bar{B}_q(1), b \in \bar{B}_q(2)\}.$$

It is easy to check that C is a code in $\bar{B}_q(1) \cup \bar{B}_q(2)$ that is capable of correcting single deletions with $|C| = \frac{(2q)^2(2q+2)}{4}$. Because $\left|\bar{B}_q(1) \cup \bar{B}_q(2)\right| = |B_{2q}|$ and $\bar{B}_q(1) \cap \bar{B}_q(2) = \emptyset$, this proves the theorem. $\square$

Combining Theorem 4.3 and 4.5 yields the following theorem.

Theorem 4.6. *There exists a code C in B_q^4 that is capable of correcting single deletions with $|C| = N(4,q,1) = \frac{q^2(q+2)}{4}$ for $q = 6m$ $(m \geq 1)$.*

Combining Theorems 3.5 and 4.6, we finally obtain the following.

Theorem 4.7. *For any even q, $N(4,q,1) = \frac{q^2(q+2)}{4}$.*

5 Perfect codes

In this section, we modify our construction of optimal codes slightly, and construct an optimal perfect code in B_q^4 when q is even.

We start with simple definitions. Recall that a code C in B_q^n is an s-covering of B_q^n from below if $\lfloor C \rfloor_s = \lfloor B_q^n \rfloor_s$ [6]. A code C in B_q^n that is capable of correcting s deletions is called a perfect code that is capable of correcting s deletions if C is an s-covering of B_q^n from below. For brevity a perfect code in B_q^n that is capable of correcting single deletions will be referred to as a perfect code.

Levenshtein [6] showed that there exists a perfect code in B_q^4 of cardinality $\frac{q^3+q^2+2q}{4}$ for any even q. Note that this code is not optimal. In previous sections, we have constructed an optimal code C in B_q^4 for any even q. By counting the cardinality of $\lfloor C \rfloor_1$, one can show that it cannot be a perfect code (for example,

the optimal code in Table 2 is not perfect). However, the construction of an optimal code can be modified to obtain an optimal perfect code as follows.

Suppose that C is an optimal code in B_q^4 for an even q which is constructed using the method in Section 2. Decompose C into subcodes C_1, C_2, and C_4, where $C_i = \{x \in C \bigm| |\lfloor x \rfloor_1| = i\}$ for $i \in \bar{B}_2$ or $i = 4$. From the structure of C, the codes C_1 and C_2 are as follows:

$$\begin{cases} C_1 = \{(a, a, a, a) \mid a \in B_q\}, \\ C_2 = \{(a, a, b, b) \mid \{a, b\} \subset B_q \text{ and } a \neq b\}. \end{cases}$$

Note that $|C_1| = q, |C_2| = q(q-1)$, and that $|C_4| = \frac{q^2(q+2)}{4} - (q + q(q-1))$. Now we modify the subcode C_2 to make C_2' as follows:

$$\begin{aligned} C_2' = &\left(C_2 \setminus \left\{(2t, 2t, 2t+1, 2t+1) \mid t \in B_{\frac{q}{2}-1}\right\}\right) \\ &\cup \left\{(2t, 2t+1, 2t, 2t+1) \mid t \in B_{\frac{q}{2}-1}\right\}. \end{aligned}$$

It can be easily verified that $C' = C_1 \cup C_2' \cup C_4$ is a code that is capable of correcting single deletions with $|C| = |C'|$. Note that $|C_2'| = |C_2|$ and $|C_2' \setminus C_2| = \frac{q}{2}$. For each $x \in C'$, the following relations arise after single deletions:

$$\begin{cases} |\lfloor x \rfloor_1| = 1 \ (x \in C_1), \\ |\lfloor x \rfloor_1| = 2 \ (x \in C_2), \\ |\lfloor x \rfloor_1| = 4 \ (x \in C_2' \setminus C_2), \\ |\lfloor x \rfloor_1| = 4 \ (x \in C_4). \end{cases}$$

Computing the cardinality of $\lfloor C' \rfloor_1$ yields:

$$\begin{aligned} |\lfloor C' \rfloor_1| &= |\lfloor C_1 \rfloor_1| + |\lfloor C_2' \rfloor_1| + |\lfloor C_4 \rfloor_1| \\ &= q + 2\left(q(q-1) - \tfrac{q}{2}\right) + 4\tfrac{q}{2} + 4\left(\tfrac{q^2(q+2)}{4} - q - q(q-1)\right) \\ &= q^3. \end{aligned}$$

Hence C' is an optimal perfect code.

Therefore we obtain the following theorem.

Theorem 5.1. *For any even q, we can construct an optimal perfect code in B_q^4 that is capable of correcting single deletions.*

Acknowledgement. The authors would like to thank to Dr. Vladimir Levedev for introducing this problem to them.

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